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Self-similar strong shocks in an exponential medium

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The self-similar one-dimensional propagation of a strong shock wave in a medium with exponentially varying density and ray-tube area is studied, using the Eulerian approach of Sedov. Conservation integrals analogous to Sedov's are obtained, with the expression for the Lagrangian variable. Calculated results are compared with the predictions of the CCW (Chisnell, Chester and Whitham) approximation. It was found that, in contrast to the implosion case, the propagation parameter from the CCW approximation is in error by 15 % or more.

1. Introduction

In connection with the problem of predicting the behaviour of strong explosions in the atmosphere, the laws governing the propagation of strong shock waves in media with exponentially varying density are of interest (see Zel'dovich & Raizer 1966, chapter XII, part 5). Plane rising and descending shocks have been investigated by Raizer (1963, 1964), using a Lagrangian formulation. The purposes of the present paper are: (*a*) to extend the analysis of Raizer to include curved shocks with exponentially varying ray-tube area, (*b*) to carry out the analysis in Sedov's Eulerian formulation, with the conservation integrals analogous to Sedov's exhibited, (*c*) to obtain numerical results over a range of the pertinent parameters, and (*d*) to compare these results with those of the CCW approximation.

The CCW approximation has been developed by Chisnell (1955, 1957), Chester (1954, 1960) and Whitham (1958), with some contributions from others. This approximation has shown phenomenal accuracy with respect to the implosion problem. In the context of the present problem the approximation is inappropriate for shock waves descending in an exponential atmosphere, and no comparison is attempted. For rising shocks the approximation is appropriate, but the accuracy turns out not to be comparable with that for the implosion problem.

Zel'dovich & Raizer (1966) suggest that the Lagrangian formulation is as convenient as the Eulerian or more so, for problems of this type. This writer disagrees. The basic differential equation to be solved numerically is in a non-analytic form in the Lagrangian formulation, but is in rational form in the Sedov formulation. The Lagrangian analogue of the basic equation (2.9) or (2.16) below involves terms with various irrational exponents. Although the question is primarily one of taste, the rational form is more convenient in numerical computation.

2. Basic theory

The gas is assumed to be a calorically perfect gas with adiabatic exponent γ. The gas is initially at rest at zero temperature and pressure under no body force, with a density distribution given by

$$\rho_\infty = \rho_0 e^{\beta x}. \tag{2.1}$$

The motion is considered to be in a channel or tube of cross-sectional area given by

$$A = A_0 e^{k\beta x}. \tag{2.2}$$

The quantity β^{-1} is the characteristic scale length for the density distribution, corresponding to the scale height in an atmosphere. We take β to be positive, with the density increasing in the direction of increasing x. In an atmosphere, the corresponding x-axis is directed downward.

The shock wave is located at $x = X(t)$, and the motion of the shock is assumed to follow the law

$$X = \alpha\beta^{-1} \ln |t|, \tag{2.3}$$

where α is the propagation or similarity parameter of the problem, equal to the α of Raizer. The velocity of the shock is then

$$\dot{X} = \alpha\beta^{-1}t^{-1}. \tag{2.4}$$

The similarity variable ξ is defined by

$$\xi = \beta(x-X); \tag{2.5}$$

it is negative behind the shock if $\dot{X}$ is positive, and positive behind the shock if $\dot{X}$ is negative (with negative t). The time t is positive and approaches $+\infty$ if $\dot{X}$ is positive, and is negative and approaches 0 if $\dot{X}$ is negative. These assumptions are suggested by dimensional analysis, using the fact that β^{-1} is the only dimensional parameter in the problem known in advance.

The dependent variables, velocity u, density ρ and pressure p, are taken to have the self-similar forms

$$u = \alpha\beta^{-1}t^{-1}V(\xi), \tag{2.6a}$$

$$\rho = \rho_0 |t|^\alpha R(\xi), \tag{2.6b}$$

$$p = \rho_0 \alpha^2\beta^{-2} |t|^{\alpha-2} P(\xi). \tag{2.6c}$$

The approach is closely analogous to that of Sedov (1957, chapter IV). The one-dimensional equations for the conservation of mass, momentum, and entropy yield directly

$$V' + (V-1)R^{-1}R' = -1-kV, \tag{2.7a}$$

$$(V-1)V' + R^{-1}P' = \alpha^{-1}V, \tag{2.7b}$$

$$(V-1)(P^{-1}P' - \gamma R^{-1}R') = 2\alpha^{-1} + (\gamma-1), \tag{2.7c}$$

where the primes indicate differentiation with respect to ξ.

Following Sedov, we introduce the variable z defined by

$$a^2 = \alpha^2\beta^{-2}t^{-2}z(\xi), \quad z = \gamma P/R. \tag{2.8}$$

The quantity a is the speed of sound, and the variable z is a reduced squared speed of sound. Elimination of ξ, P and R from (2.7) then yields the basic differential equation

$$\frac{d\ln z}{d\ln(1-V)} = -\frac{[2-(\gamma-1)k\alpha V](1-V)^2+(\gamma-1)V(1-V)-[2-(\gamma-1)\kappa]z}{V(1-V)+(\kappa-k\alpha V)z}. \tag{2.9}$$

The parameter κ is given by

$$\kappa = \frac{2-\alpha}{\gamma}, \quad \alpha = 2-\gamma\kappa, \tag{2.10}$$

and may be used to replace α. The bracket multiplying z in the numerator of (2.9) may be written in the forms

$$2-(\gamma-1)\kappa = \frac{2+(\gamma-1)\alpha}{\gamma} = \alpha+\kappa. \tag{2.11}$$

The remaining variables are obtainable from a solution $z(V)$ of (2.9) through (2.8) and quadratures of

$$\frac{\alpha dV}{d\xi} = \frac{V(1-V)+(\kappa-k\alpha V)z}{z-(1-V)^2}, \tag{2.12}$$

$$\frac{d\ln(1-V)R}{d\xi} = \frac{1+kV}{1-V}. \tag{2.13}$$

A direct but detailed manipulation of (2.9) with either (2.7) or (2.12) and (2.13) yields a first integral for the system

$$(1-V)^{2+(\gamma-1)\alpha}R^{2-(\gamma-1)k\alpha}z^{\alpha(1+k)}e^{[2+(\gamma-1)\alpha]k\xi} = const. \tag{2.14}$$

A method is to find the proper linear combination of the logarithmic derivatives of z, $(1-V)$ and R with respect to ξ so that the right-hand side is independent of z and V, i.e. is a constant. A quadrature then yields the logarithm of (2.14). This integral is equivalent to that obtained by eliminating the Lagrangian variable between the two general algebraic integrals of Sedov or the analogous two of Raizer (1964). Thus only (2.12) needs to be integrated, with (2.14) then serving to give R.

A convenient transformation simplifies the basic equation (2.9) somewhat. This transformation is

$$z = (1-V)\zeta, \tag{2.15}$$

and leads to the equation

$$\frac{d\ln\zeta}{d\ln(1-V)} = -\frac{[2-(\gamma-1)k\alpha V](1-V)+\gamma V-\alpha(1+kV)\zeta}{V+(\kappa-k\alpha V)\zeta} \tag{2.16}$$

in place of (2.9). The right-hand side of (2.12) simplifies somewhat, to give

$$\frac{\alpha dV}{d\xi} = \frac{V+(\kappa-k\alpha V)\zeta}{\zeta-(1-V)}. \tag{2.17}$$

An analogous transformation is available in Sedov's case.

Of the several critical points of the basic equation (2.9) or (2.16), the most important are those that lie on the curve $z = (1-V)^2$ or $\zeta = 1-V$. These are located at V_{cr}, ζ_{cr}, with

$$k\alpha V_{cr}^2 + (1-\kappa-k\alpha)V_{cr} + \kappa = 0, \tag{2.18a}$$

$$\zeta_{cr} = 1 - V_{cr}. \tag{2.18b}$$

If $k = 0$, there is a single point of this type at $\zeta_{cr} = 1-V_{cr} = (1-\kappa)^{-1}$. Such points are important because the problem is generally of the type termed by Zel'dovich & Raizer a self-similar problem of the second kind. In problems of this type the propagation exponent or parameter (α in this instance) must be determined by the requirement that the solution pass through a saddle-point on the curve $z = (1-V)^2$ or its equivalent. This curve is a characteristic of the original hyperbolic equation; hence, the possibility of singularities arises.

Boundary conditions in front of the shock are simply $V_\infty = 0$, $R_\infty = 1$, $P_\infty = 0$, $z_\infty = 0$ and $\zeta_\infty = 0$. The conditions immediately behind the shock are then

$$V_s = P_s = \frac{2}{\gamma+1}, \tag{2.19a}$$

$$R_s = (1-V_s)^{-1} = \frac{\gamma+1}{\gamma-1}, \tag{2.19b}$$

$$z_s = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}, \tag{2.19c}$$

$$\zeta_s = \frac{2\gamma}{\gamma+1}. \tag{2.19d}$$

Boundary conditions at an infinite distance behind the shock depend upon the particular problem. With descending shocks (t positive) with $k \leqslant 0$, $\xi = -\infty$ and $\zeta = 0$ at $V = -\infty$. With descending shocks with $k > 0$ the self-similar solution cannot extend to $\xi = -\infty$. With ascending shocks (t negative), k must be greater than some critical value (perhaps $-\gamma^{-1}$), and $\xi = +\infty$, $\zeta = 0$ at $V = 0$. In any of these cases the solution must cross the line $\zeta = 1-V$ at a saddle-point, and this condition is the condition that determines the value of α.

Self-similar problems of the second kind are thoroughly discussed in chapter XII of Zel'dovich & Raizer (1966), with several examples treated. In all these problems insufficient information is available in advance to give the value of the characteristic exponent or parameter directly. In all these problems an actual solution is initially not self-similar, and approaches a self-similar one as time increases. They are all governed by the fact that the boundary conditions dictate that the solution must cross the line $\zeta = 1-V$ or equivalent, and that if the numerator of (2.17) or a corresponding equivalent is not zero at the crossing point, the point corresponds to a gasdynamic 'limiting line' and the solution is physically unacceptable. Thence comes the saddle-point condition, which determines the similarity parameter. The line $\zeta = 1-V$ corresponds to a characteristic of the original hyperbolic set of equations in each case.

Certain anomalies in the behaviour of the integrals for the total momentum and energy in a ray tube are explained in chapter XII, part 4 of Zel'dovich &

Raizer (1966), in terms of the difference between an actual solution and its limiting self-similar form. This explanation applies in all essentials to the present problem. For example, the integral for the energy in the self-similar solution at any instant diverges. This divergence appears in the particular domain where an actual solution is very different from the self-similar one, so that in any particular actual case the energy is finite. The reader is referred to Zel'dovich & Raizer for an extensive discussion of this and related points.

To complete the presentation in close analogy with Sedov's, we next present general conservation integrals, together with certain solutions corresponding to constant momentum or energy in a ray tube. These solutions are not, in general, physically acceptable, as they usually violate the saddle-point condition. They are important primarily in serving as limiting or bounding solutions.

3. Conservation integrals and special solutions

The total mass in a ray tube above a certain point x is given by

$$\mathscr{M} = \int_{-\infty} \rho A \, dx = A_0 \rho_0 \beta^{-1} |t|^{(1+k)\alpha} \eta, \tag{3.1}$$

where $\eta(\xi)$ is a reduced Lagrangian variable defined by

$$\eta = \int_{-\infty} R\, e^{k\xi} \, d\xi. \tag{3.2}$$

Integrals without upper limit are indefinite integrals equal to zero at the lower limit. The variable η defined for the case $k = 0$ is used as the independent variable by Raizer (1963, 1964), with a differential equation for $P(\eta)$ (in terms of a variable p_* or f) in place of our equation (2.9) or (2.16).

The quantity $\mathscr{M}$ is a physical Lagrangian variable, and the quantity $p\rho^{-\gamma}$ must be a function of $\mathscr{M}$ alone. Introducing the forms of (2.6) and requiring the function to be consistent in powers of $|t|$ leads directly to

$$\eta^{\{(\gamma-1)\alpha+2\}/(1+k)\alpha} P R^{-\gamma} = \mathit{const.} \tag{3.3}$$

This result is directly analogous to Sedov's entropy integral, and was given by Raizer for $k = 0$.

Conservation of mass requires that the time derivative of $\mathscr{M}$ with ξ held fixed be balanced by the mass flow across the plane ξ = constant. This principle leads to the relation

$$(1+k)\,\eta = R(1-V)\, e^{k\xi}, \tag{3.4}$$

the analogue of Sedov's mass conservation integral, also given by Raizer for $k = 0$. Elimination of η between (3.3) and (3.4) gives the integral (2.14) above.

For momentum conservation we introduce the integral

$$\mathscr{I} = \int_{-\infty} \rho u A \, dx = A_0 \rho_0 \alpha \beta^{-2} |t|^{\alpha(1+k)} t^{-1} I, \tag{3.5}$$

where $I(\xi)$ is defined

$$I = \int_{-\infty} R V \, e^{k\xi} \, d\xi. \tag{3.6}$$

The total momentum above point ξ is equal to $\mathscr{I}(\xi, t)$. Conservation of momentum, applied to a calculation of $d\mathscr{I}/dt$ with ξ fixed, leads to

$$[(1+k)\alpha-1]I+\alpha e^{k\xi}[RV(V-1)+P]|_{-\infty}-\alpha k\int_{-\infty} P e^{k\xi}\,d\xi = 0. \tag{3.7}$$

The time derivative of $\mathscr{I}$ here is equated to the flux of momentum past the point ξ plus the pressure force exerted on the ray tube. The relation (3.7) may also be obtained from the equations of motion. The case

$$\alpha = \frac{1}{1+k} \tag{3.8}$$

is termed the constant-momentum case, as it corresponds to $d\mathscr{I}/dt = 0$. If $k = 0$ a specific result is obtained,

$$P+RV(V-1) = const., \tag{3.9}$$

with $\alpha = 1$. The solution for which this constant is zero is

$$\frac{z}{1-V} = \zeta = \gamma V, \tag{3.10}$$

and is of particular interest because it fits the shock conditions (2.19).

For energy conservation we introduce the integral

$$\mathscr{E} = \int_{-\infty}\left(\frac{p}{\gamma-1}+\frac{\rho u^2}{2}\right)A\,dx = A_0\rho_0\alpha^2\beta^{-3}\,|t|^{\alpha(1+k)-2}E, \tag{3.11}$$

where $E(\xi)$ is defined

$$E = \int_{-\infty}\left(\frac{P}{\gamma-1}+\frac{RV^2}{2}\right)e^{k\xi}\,d\xi. \tag{3.12}$$

The total energy in a ray tube above point ξ is equal to $\mathscr{E}(\xi, t)$. Conservation of energy, applied to a calculation of $d\mathscr{E}/dt$ with ξ fixed, leads to

$$[(1+k)\alpha-2]E+\alpha e^{k\xi}\left[\left(\frac{P}{\gamma-1}+\frac{RV^2}{2}\right)(V-1)+PV\right]\Bigg|_{-\infty} = 0. \tag{3.13}$$

This relation may also be derived from the equations of motion. The case

$$\alpha = \frac{2}{1+k} \tag{3.14}$$

is termed the constant-energy case, in which we obtain

$$\left(\frac{P}{\gamma-1}+\frac{RV^2}{2}\right)(V-1)+PV = e^{-k\xi}\cdot const. \tag{3.15}$$

The solution for which the constant is zero is

$$\frac{z}{1-V} = \zeta = \frac{\gamma(\gamma-1)\,V^2}{2(\gamma V-1)}, \tag{3.16}$$

and again fits the shock conditions (2.19). This solution is the direct analogue of Sedov's noted solution for the point explosion.

A special type of solution appears which does not emerge from one of the conservation principles. This solution has $\zeta = const$, and this constant is chosen to fit (2.19). The solution is thus

$$\frac{z}{1-V} = \zeta = \frac{2\gamma}{\gamma+1}. \tag{3.17}$$

One solution of this type, for which the author is indebted to the referee, is

$$V = \frac{2}{\gamma+1} = const, \quad \alpha = \frac{\gamma-3}{\gamma-1}, \quad k = \frac{\gamma+1}{\gamma-3}, \tag{3.18}$$

with R and P given by

$$R = R_s \exp\{(\gamma+1)\xi/(\gamma-1)\}, \quad P = P_s \exp\{(\gamma+1)\xi/(\gamma-3)\}. \tag{3.19}$$

If the variable V is not constant, examination of (2.16) shows that $(\gamma-1)k$ must be zero. Two cases thus appear for this special solution. The first is characterized by

$$k = 0, \quad \gamma = 2, \quad \alpha = \tfrac{3}{2}, \quad \zeta = \tfrac{4}{3}. \tag{3.20}$$

This case was found by Raizer (1963), and is a solution corresponding to a descending shock. The other case is characterized by

$$k = -\tfrac{1}{2}, \quad \gamma = 1, \quad \alpha = 2, \quad \zeta = 1, \tag{3.21}$$

with the relation (3.8) valid. It may be considered a limiting case of a descending shock, in the limit $\gamma \to 1$.

4. The limiting case $\gamma \to 1$

The limiting case $\gamma \to 1$ is singular whenever a shock is present. In the limit $V_s = 1$, which means that $u = \dot{X}$ immediately behind the shock. The density function $R_s \to \infty$ behind the shock, and the behaviour of R near the shock approaches that of a delta function. Farther behind the shock the function R drops to values that approach zero in the limit. The total moving mass is concentrated in an infinitesimal 'shock' layer immediately behind the shock.

The region behind this shock layer is a constant pressure region, characterized by a single pressure $p(t)$. This back pressure may be zero. In this region of negligible density our equations may define a definite velocity, as in the solution of (3.21). Behind the shock layer this solution is meaningless. In the full limit $\gamma \to 1$ the solution may cross the line $\zeta = 1 - V$ at an ordinary point. The anomaly at the crossing point is spurious, because it is for a flow with zero density.

The solution in this limiting case may be obtained directly from the dynamics of the shock layer. Such a theory is a simple case of the Newtonian theory of hypersonic flow, and the limiting approximation is sometimes referred to as the snowplow approximation. In our problem, we must make a sharp distinction between the cases $k > -1$ and $k < -1$. Here we assume $k > -1$ and simply note that, if $k < -1$, the roles of rising and descending shocks are reversed. The case $k = 1$ is a simple special case, but one for which the self-similar forms are not of the type considered in this paper.

For descending shocks we assume that the entire mass between the shock and $x = -\infty$ is concentrated in the shock layer. The momentum of the layer is $\mathscr{I} = \mathscr{M}\dot{X}$, and conservation of momentum gives

$$\frac{d}{dt}(\mathscr{M}\dot{X}) = pA. \tag{4.1}$$

The mass $\mathscr{M}$ is given by (3.1) with $\eta = (1+k)^{-1}$, and $\dot{X}$ and A are given by (2.2) to (2.4), with $x = X$. The pressure p_s behind the shock is simply $\rho_\infty \dot{X}^2$. For self-similarity we must require that p/p_s be a constant. The relation (4.1) leads to a condition giving the parameter α,

$$\alpha = \frac{1}{(1+k)(1-p/p_s)}. \tag{4.2}$$

Note that with $p/p_s = 0$, we have the constant-momentum case of (3.8). With $p/p_s = \frac{1}{2}$, we have the constant-energy case of (3.14). This limiting behaviour corresponds with that of Sedov's solutions in the limit $\gamma \to 1$.

In the case of an ascending shock we must assume that a fixed mass $\mathscr{M}_0$ has been concentrated in the shock layer, and that the total mass $\mathscr{M}$ above the shock is negligible compared with $\mathscr{M}_0$. The relation (4.1) holds with the sign on p changed. We obtain

$$\begin{aligned} p &= A^{-1}\mathscr{M}_0\alpha\beta^{-1}t^{-2} \\ &= A_0^{-1}\mathscr{M}_0\alpha\beta^{-1}|t|^{-2-k\alpha}. \end{aligned} \tag{4.3}$$

This pressure is large compared with p_s, but may be compared with a reference pressure given by

$$p_{\mathrm{ref}} = \mathscr{M}_0 A^{-1}\beta\dot{X}^2. \tag{4.4}$$

Combining (4.3) and (4.4) gives

$$\frac{p}{p_{\mathrm{ref}}} = \alpha^{-1}, \tag{4.5}$$

and shows that in the limit $p \to 0$ with $\dot{X}$ kept finite, the parameter $\alpha \to \infty$.

5. The CCW approximation

The method given by Whitham (1958) for establishing the CCW approximation is to write the characteristic equation for the characteristics moving in the same direction as the shock and to apply this equation in terms of the variables evaluated immediately behind the shock. In our case with a rising shock, the characteristic equation is

$$dp - \rho a\, du + \frac{\rho u a^2}{u-a}\frac{dA}{A} = 0. \tag{5.1}$$

We express p_s, u_s, ρ_s, $a_s^2 = \gamma p_s/\rho_s$ and $A_s = A(X)$ as functions of time, using (2.2) to (2.6) and (2.19), keeping in mind that u and t are negative. These quantities, are then substituted into (5.1). The result is an expression for $\alpha(k,\gamma)$, which we give in the form

$$\alpha^{-1} = \left(2+\left[\frac{2\gamma}{\gamma-1}\right]^{\frac{1}{2}}\right)^{-1}\left\{1+\gamma k\left(1+\left[\frac{\gamma(\gamma-1)}{2}\right]^{\frac{1}{2}}\right)^{-1}\right\}. \tag{5.2}$$

Note that α^{-1} is linear in k. We may write

$$2\alpha^{-1} = K_0 + K_1 k, \tag{5.3}$$

where K_0 and K_1 depend only upon γ.

The quantity K_1 is the same as K_∞ in equation (11) of Whitham (1958), while $\frac{1}{2}K_0$ is the same as β in equation (33) of the same paper. In the limit $k \to \infty$, $\alpha \to 0$, the quantity $\frac{1}{2}K_1$ is to be interpreted as $(\alpha k)^{-1}$, and it is this quantity in this limit that agrees so phenomenally with exact calculations for the implosion problem. Comparisons with exact calculations for $\alpha(k, \gamma)$ in our problem are given below.

6. Calculations for rising shocks

Integrations of the differential equation (2.16) subject to the boundary conditions (2.19) and the saddle-point condition were carried out for a number of choices of the parameters by V. Sagherian of the Stanford Research Institute. For rising shocks these calculations included a number of values of k at $\gamma = 1{\cdot}4$, a number of values of γ at $k = 0$, and a smaller number of values of γ at $k = \pm 0{\cdot}1$. The quantity $\alpha > 2$ in these cases, and κ is negative.

The method of calculation is about the same for all problems of the second kind. First, the local solutions in the neighbourhoods of the saddle points and the shock point (2.19) are studied. This study gives information on the local slopes of the solution curve at these points on the (V, ζ)-plane. For a number of choices of α, the solution is started on the correct branch leaving the appropriate saddle-point and is continued to V_s. The process is repeated for values of α between those bracketing the correct value of ζ_s, until the error in ζ_s is small. An accurate value of α is then obtained by interpolation.

We present here the values of α obtained, in the form

$$2\alpha^{-1} = K(k, \gamma) = K_0(\gamma) + K_1(\gamma) k + O(k^2) \tag{6.1}$$

in analogy with (5.3). The results for $\gamma = 1{\cdot}4$ are presented in table 1, with the corresponding results according to the CCW approximation.

For varying γ we choose integral values of the quantity $2/(\gamma - 1)$, and present K_0 and K_1 as defined by (6.1) in table 2, with the corresponding values according to the CCW approximation.

Raizer (1964) reported $\alpha = 4{\cdot}90$ and $6{\cdot}48$ for the cases $\gamma = \frac{5}{3}$ and $\frac{6}{5}$ with $k = 0$, in accord with our values of K_0 in table 2. He also reported that at $t = 0$, at the moment the shock reaches $x = -\infty$, each particle in the flow field has moved upward a distance $\delta\beta^{-1}$, with $\delta = 4{\cdot}57$ for $\gamma = \frac{5}{3}$ and $\delta = 7{\cdot}50$ for $\gamma = \frac{6}{5}$. We have not made the corresponding calculation in our cases.

Curves for the parameters are presented in an accompanying paper (Hayes 1968, figures 1 and 2). It may be observed either from the tables here or the figures in the other paper that the CCW approximation is in error by some 15 % or 20 % for K_0. For K_1 the error is less, and is zero for a value of γ of about $1{\cdot}21$. With $\gamma = 1{\cdot}4$ the error in K is large for negative k (as might be expected), and drops to about 7 % at $k = 1$. The phenomenal agreement found in the implosion case is not found here, but there is a strong indication that the agreement should again be excellent in the limit $k \to \infty$, with an exponentially varying ray-tube area and constant density in the undisturbed flow.

Sakurai (1960) has considered a one-dimensional flow without area change in which the shock approaches a distinct edge, with the undisturbed density a power of the distance from the edge. Calculations were carried out for the power equal

k	K	K_{CCW}
−0·5	0·1569	0·2335
−0·4	0·1998	0·2729
−0·3	0·2414	0·3123
−0·2	0·2838	0·3517
−0·1	0·3257	0·3911
0	0·3673	0·4305
0·1	0·4086	0·4699
0·2	0·4496	0·5093
0·3	0·4907	0·5487
0·4	0·5314	0·5881
0·6	0·6126	0·6670
0·8	0·6933	0·7458
1·0	0·7726	0·8246

TABLE 1. Parameter K for rising shocks, with $\gamma = 1{\cdot}4$

γ	$\dfrac{2}{\gamma-1}$	K_0	$K_{0,\mathrm{CCW}}$	K_1^*	$K_{1,\mathrm{CCW}}$
2	2	0·438	0·500	—	0·500
$\frac{5}{3}$	3	0·408	0·472	—	0·451
$\frac{3}{2}$	4	0·386	0·449	0·447	0·418
$\frac{7}{5}$	5	0·367	0·430	0·414	0·394
$\frac{4}{3}$	6	0·352	0·414	0·388	0·377
$\frac{9}{7}$	7	0·339	0·400	—	0·360
$\frac{5}{4}$	8	0·328	0·387	0·352	0·347
$\frac{11}{9}$	9	0·318	0·376	—	0·336
$\frac{6}{5}$	10	0·309	0·366	0·324	0·326

* K_1 was only computed for five values of γ.

TABLE 2. Parameters for rising shocks, with $k = 0$

to $\frac{1}{2}$, 1 and 2, and for $\gamma = \frac{5}{3}, \frac{7}{5}$ and $\frac{6}{5}$. Values of K_0 were obtained which lie between those of our calculations for $k = 0$ and the corresponding CCW values. Our case $k = 0$ corresponds to the limiting case of Sakurai's in which the power approaches infinity. Sakurai's and related results are discussed by Zel'dovich & Raizer (1966).

The CCW approximation would appear to be generally better for decreasing area than for decreasing density, and it is not clear why this should be so. That the approximation should be better with power-law profiles than with exponential profiles is not unexpected.

7. Calculations for descending shocks

Our calculations for descending shocks were carried out only for $k = 0$, with a number of choices of γ. Here $\alpha < 2$ and $\kappa > 0$ in the cases considered. In the calculation, the shock point and the saddle point are considerably farther apart on the (V, ζ)-plane than in the case of a rising shock, with the result that the calculated values of α are more sensitive to the accuracy of the calculation. Our results are in table 3.

The result for $\gamma = 2$ is Raizer's exact solution corresponding to (3.18) above. Raizer (1963) also reported the value $\alpha = 1{\cdot}345$ for $\gamma = \frac{5}{4}$, in slight disagreement with our value 1·338.

γ	$\dfrac{2}{\gamma-1}$	α
2	2	1·500
$\frac{5}{3}$	3	1·450
$\frac{3}{2}$	4	1·417
$\frac{7}{5}$	5	1·392
$\frac{4}{3}$	6	1·369
$\frac{9}{7}$	7	1·351
$\frac{5}{4}$	8	1·338
$\frac{11}{9}$	9	1·324

TABLE 3. Parameter α for descending shocks, with $k = 0$.

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